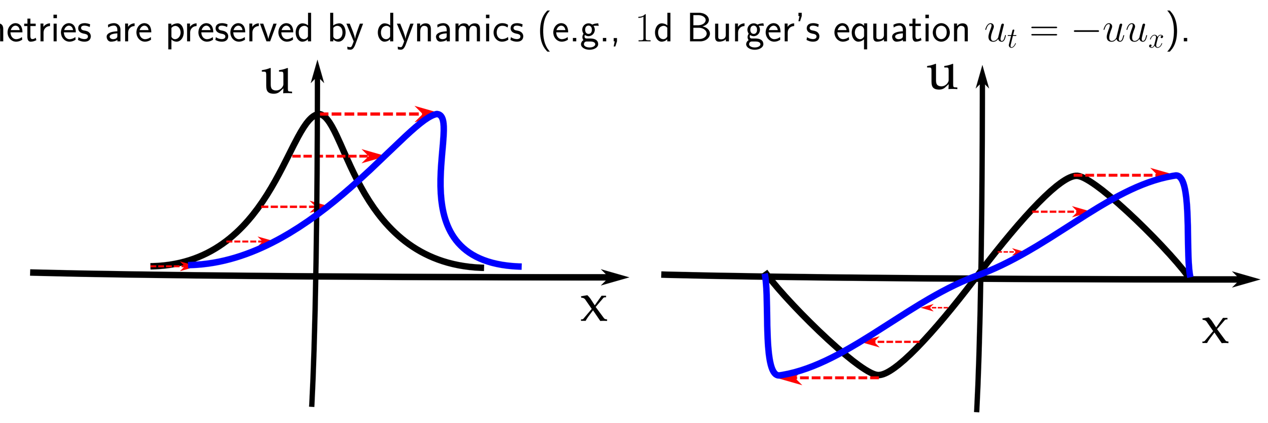
Invariant Subspaces of Channel Flow

Pratik Aghor and John F. Gibson

Integrated Applied Mathematics Program, Univeristy of New Hampshire, USA

Motivation

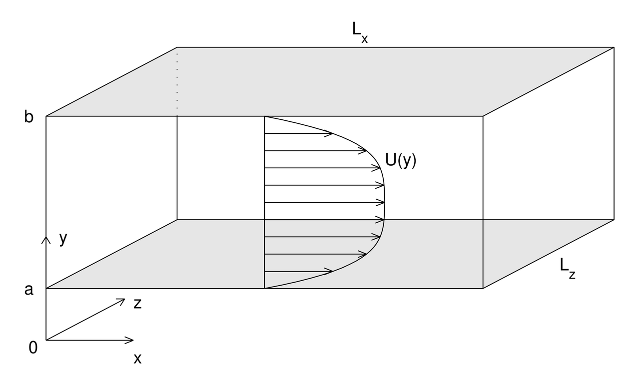
- Symmetries influence dynamics, determine kinds of allowable solutions.
- Certain symmetries are preserved by dynamics (e.g., 1d Burger's equation $u_t = -uu_x$).



Even initial conditions (IC's) lose even-ness, odd IC's remain odd under time evolution.

- Symmetries reduce the dimensionality of the search space - useful in numerical computations of invariant solutions.
- Determine and classify the subgroups of the Plane Poiseuille Flow (PPF) symmetry group, to aid in the search for and classification of invariant solutions of PPF [Aghor and Gibson, 2022].

Plane Poiseuille Flow



PPF schematic.

- Base state: $[U, V, W](x, y, z) = [1 - y^2, 0, 0]$.
- $\boldsymbol{u}_{\text{tot}} = \boldsymbol{U} + \boldsymbol{u}$, $p_{\text{tot}} = P + p$, $\nabla p_{\text{tot}} = P_x \boldsymbol{e}_x + \nabla p$.
- Constant bulk velocity constraint $\langle \boldsymbol{u}_{\text{tot}} \rangle_\Omega = \langle \boldsymbol{U} \rangle_\Omega = U_{\text{bulk}} = 2/3$ and $\langle \boldsymbol{u} \rangle_\Omega = 0$, with $\langle \boldsymbol{u}_{\text{tot}} \rangle_\Omega = \langle \boldsymbol{U} \rangle_\Omega = U_{\text{bulk}} = 2/3$ and $\langle \boldsymbol{u} \rangle_\Omega = 0$, where $\langle \cdot \rangle_\Omega = \frac{1}{\text{vol}(\Omega)} \int_\Omega \cdot \, d\Omega$.
- Base pressure P_x in Eqn.(1) is then a dynamic variable $P_x(t)$, adjusting to satisfy the bulk velocity constraint.

$$\frac{\partial \boldsymbol{u}}{\partial t} + v\boldsymbol{U'} + [\boldsymbol{U} \cdot \nabla]\boldsymbol{u} + [\boldsymbol{u} \cdot \nabla]\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u} + \left(\frac{1}{Re}U'' - P_x\right)\boldsymbol{e_x}, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{1}$$

Symmetry Operations

- Governing equations (1) symmetric under following transformations:

$$\begin{aligned} \sigma_y[u, v, w](x, y, z) &= [u, -v, w](x, -y, z), \\ \sigma_z[u, v, w](x, y, z) &= [u, v, -w](x, y, -z), \\ \tau(\ell_x, \ell_z)[u, v, w](x, y, z) &= [u, v, w](x + \ell_x, y, z + \ell_z), \end{aligned} \tag{2}$$

- A general symmetry operation γ can be written as follows [Gibson et al., 2009] -

$$\gamma[u, v, w](x, y, z) = [s_x u, s_y v, s_z w](s_x x + a_x L_x, s_y y, s_x z + a_z L_z), \tag{3}$$

- s_x and s_z can take values $\{\pm 1\}$,
- a_x and a_z denote streamwise and spanwise translations, and can take real values.
- Can write a data-structure that represents symmetry operations for PPF:

$$\begin{aligned} \sigma_y &= (1, -1, 1, 0, 0), \\ \sigma_z &= (1, 1, -1, 0, 0), \\ \tau(\ell_x, \ell_z) &= (1, 1, 1, a_x, a_z), \end{aligned} \tag{4}$$

where $a_x = \ell_x / L_x$ and $a_z = \ell_z / L_z$.

- Can define a multiplication operation for two symmetry-objects $\gamma_1 \equiv (s_{1x}, s_{1y}, s_{1z}, a_{1x}, a_{1z})$ and $\gamma_2 \equiv (s_{2x}, s_{2y}, s_{2z}, a_{2x}, a_{2z})$ -

$$\gamma_1 \cdot \gamma_2 = (s_{1x} \cdot s_{2x}, s_{1y} \cdot s_{2y}, s_{1z} \cdot s_{2z}, \text{mod}(a_{1x} + s_{1x}a_{2x}, 1), \text{mod}(a_{1z} + s_{1z}a_{2z}, 1)). \tag{5}$$

- Consistent with physical meaning of transformations.

Why Are Half-box-shifts Important?

$$\begin{aligned} \tau(\ell_x, 0)\, \sigma_y\, \tau(\ell_x, 0)\, \sigma_y\, \boldsymbol{u} &= \tau(\ell_x, 0)\, \sigma_y\, \boldsymbol{u}, \\ \tau(\ell_x, 0)\, \tau(\ell_x, 0)\, \sigma_y\, \sigma_y\, \boldsymbol{u} &= \boldsymbol{u}, \\ \tau(2\ell_x, 0)\, \boldsymbol{u} &= \boldsymbol{u}. \end{aligned} \tag{6}$$

- Thus $\tau(\ell_x, 0)\, \sigma_y$ symmetry implies x periodicity with length $2\ell_x$ and is equivalent to $\tau(L_x/2, 0)\, \sigma_y$ symmetry on a $L_x = 2\ell_x$ periodic domain.
- Only consider the group $G = \langle \sigma_y, \sigma_z, \tau_x, \tau_z \rangle$ with discrete symmetries.
- Can show that G is Abelian.

Quarter-box-shift Conjugacies

- For notational convenience, define half-box shifts in streamwise and spanwise directions as

$$\begin{aligned} \tau_x &= \tau(\tfrac{1}{2}L_x, 0) = (1, 1, 1, 0.5, 0), \\ \tau_z &= \tau(0, \tfrac{1}{2}L_z) = (1, 1, 1, 0, 0.5). \end{aligned} \tag{7}$$

- Also, $\tau_{xz} = \tau_x \tau_z = (1, 1, 1, 0.5, 0.5)$.
- We further define quarter-box-shifts as

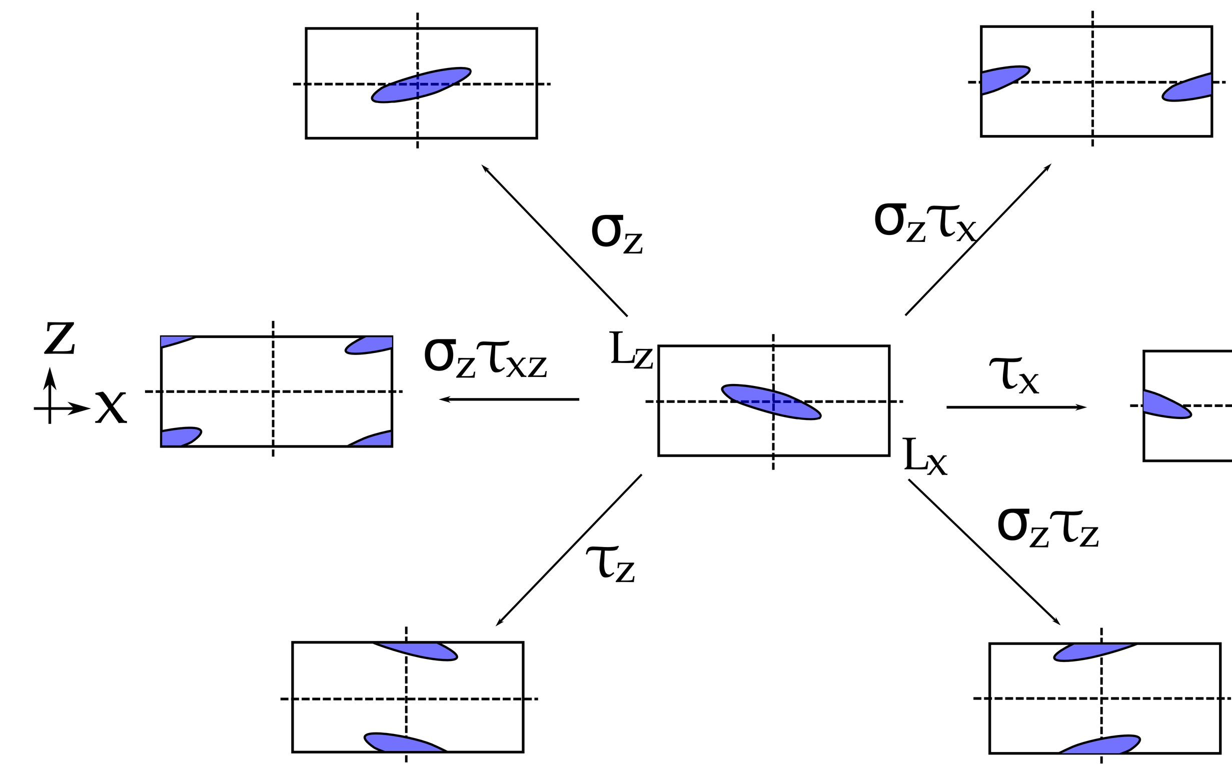
$$\begin{aligned} \tau_x^{1/2} &= \tau(\tfrac{1}{4}L_x, 0) = (1, 1, 1, 0.25, 0), \\ \tau_z^{1/2} &= \tau(0, \tfrac{1}{4}L_z) = (1, 1, 1, 0, 0.25). \end{aligned} \tag{8}$$

- Consider two subgroups of G, $\langle \sigma_z \rangle$ and $\langle \sigma_z \tau_z \rangle$.
- Can show $\langle \sigma_z \rangle \sim \langle \sigma_z \tau_z \rangle$ under quarter-box-shift conjugation, i.e., $\sigma_z \tau_z = \tau_z^{-1/2} \sigma_z \tau_z^{1/2}$.

$$\begin{aligned} \tau_z^{-1/2} \sigma_z \tau_z^{1/2} &= (1, 1, 1, 0, -0.25) \cdot (1, 1, -1, 0, 0) \cdot (1, 1, 1, 0, 0.25) \\ &= (1, 1, 1, 0, -0.25) \cdot (1, 1, -1, 0, 0.75) \\ &= (1, 1, -1, 0, 0.5) \\ &= \sigma_z \tau_z. \end{aligned} \tag{9}$$

- Quarter-box-shift conjugation partitions the invariant subgroups of G in conjugacy classes.

Example

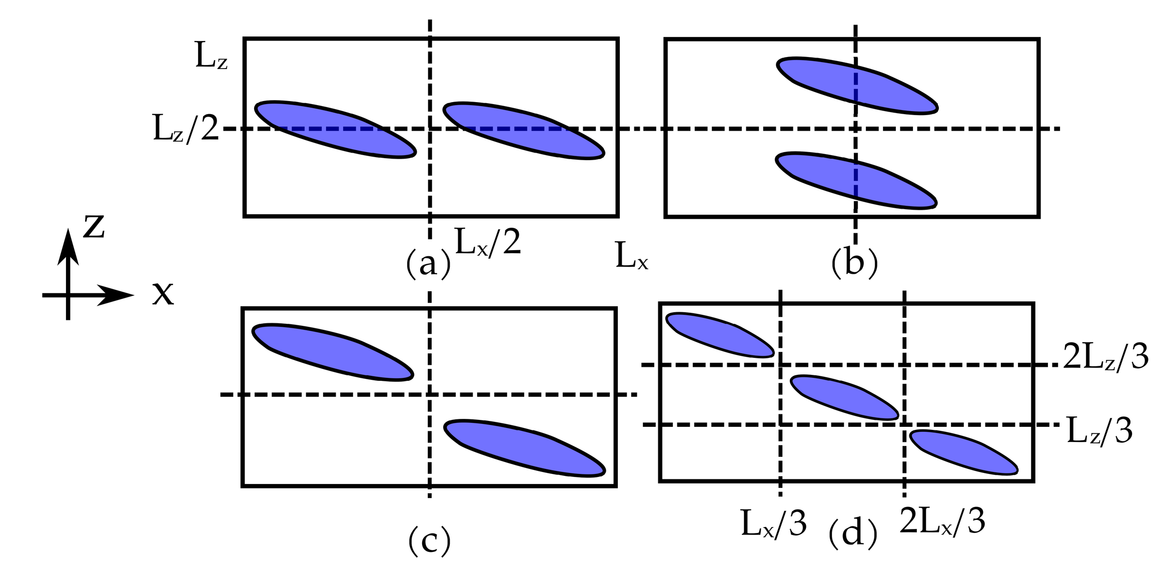


Examples of transformations in the $x - z$ plane.

Algorithm to Find Relevant Subgroups

- Find subgroups of orders $2, 4, 8$ using Lagrange's theorem/ different generators.
- Ignore subgroups where τ_x and τ_z appear as single entries in the list of generators. Periodicity on smaller domains.
- Between remaining subgroups of the same order check for quarter-box-shift conjugacies.
- Only consider one subgroup per conjugacy class.

Why Ignore τ_x and τ_z as Single Generators?



Cartoon for cases where (a) τ_x, (b) τ_z, (c) τ_{xz} and (d) $\tau(L_x/3, L_z/3)$ appear as independent generators of a subgroup. In the figures (a) and (b), we can consider smaller periodic domains of lengths $L_x/2$ and $L_z/2$, respectively. However, no such reduction is possible in the case of figures (c) and (d).

Independent Non-trivial Subgroups of G

$\langle \tau_{xz} \rangle = \{1, \tau_{xz}\}$	$\langle \sigma_y \rangle = \{1, \sigma_y\}$	$\langle \sigma_z \rangle = \{1, \sigma_z\}$
$\langle \sigma_z \tau_x \rangle = \{1, \sigma_z \tau_x\}$	$\langle \sigma_y \tau_z \rangle = \{1, \sigma_y \tau_z\}$	$\langle \sigma_y \tau_x \rangle = \{1, \sigma_y \tau_x\}$
$\langle \sigma_y \tau_{xz} \rangle = \{1, \sigma_y \tau_{xz}\}$	$\langle \sigma_{yz} \rangle = \{1, \sigma_{yz}\}$	$\langle \sigma_{yz} \tau_x \rangle = \{1, \sigma_{yz} \tau_x\}$

Table: List of independent order-2 subgroups up to quarter-box-shift conjugacy.

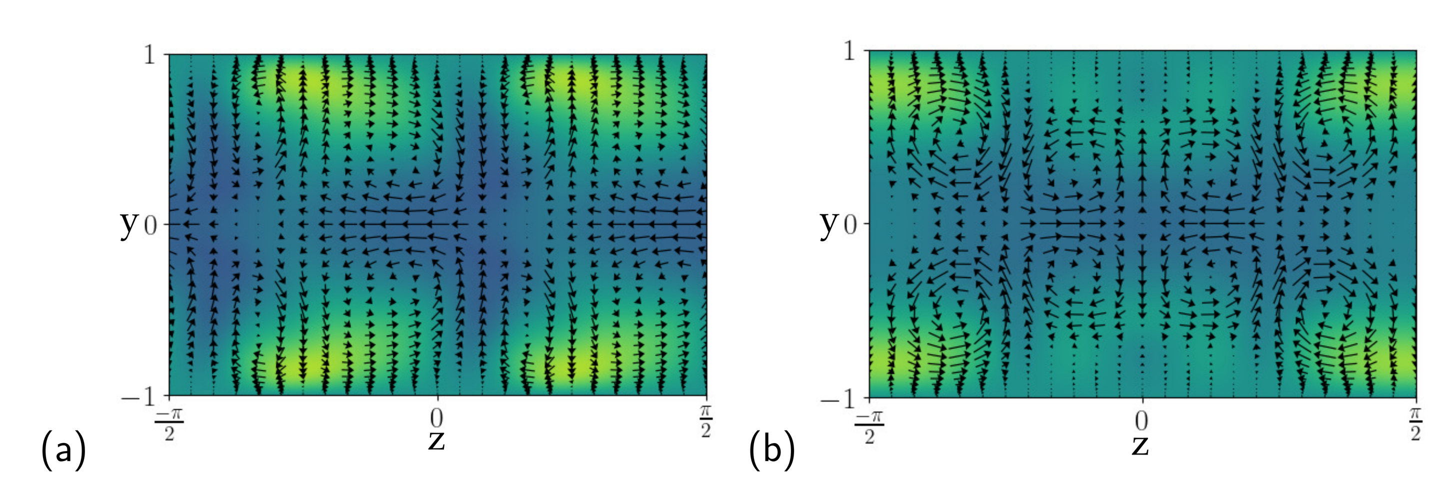
$\langle \tau_{xz}, \sigma_z \rangle$	$\langle \tau_{xz}, \sigma_y \rangle$	$\langle \tau_{xz}, \sigma_y \tau_z \rangle$
$\langle \tau_{xz}, \sigma_{yz} \rangle$	$\langle \sigma_z, \sigma_y \rangle$	$\langle \sigma_z, \sigma_y \tau_z \rangle$
$\langle \sigma_z, \sigma_y \tau_x \rangle$	$\langle \sigma_z, \sigma_y \tau_{xz} \rangle$	$\langle \sigma_z \tau_x, \sigma_y \rangle$
$\langle \sigma_z \tau_x, \sigma_y \tau_z \rangle$	$\langle \sigma_z \tau_x, \sigma_y \tau_x \rangle$	$\langle \sigma_z \tau_x, \sigma_y \tau_{xz} \rangle$

Table: List of independent order-4 subgroups up to quarter-box-shift conjugacy.

$\langle \tau_{xz}, \sigma_z, \sigma_y \rangle$
$\langle \tau_{xz}, \sigma_z, \sigma_y \tau_z \rangle$

Table: List of independent order-8 subgroups up to quarter-box-shift conjugacy.

Dynamics in Invariant Subspaces: Examples of TWs



Comparison between streamwise averaged $y - z$ cross sections of TWs with (a) $\langle \tau_{xz}, \sigma_y \rangle$ and (b) $\langle \sigma_z \tau_x, \sigma_y \rangle$ symmetries imposed. TW in (a) travels in both streamwise and spanwise directions while TW in (b) only travels in the streamwise direction. Computed using Channelflow 2.0 [Gibson et al., 2008].

References

[Aghor and Gibson, 2022] Aghor, P. and Gibson, J. F. (2022). Invariant symmetric subspaces of plane Poiseuille flow. *upnder prep.*, ??

[Gibson et al., 2008] Gibson, J. F., Halcrow, J., and Cvitanović, P. (2008). Visualizing the geometry of state space in plane Couette flow. *Journal of Fluid Mechanics*, 611:107–130.

[Gibson et al., 2009] Gibson, J. F., Halcrow, J., and Cvitanović, P. (2009). Equilibrium and travelling-wave solutions of plane Couette flow. *Journal of Fluid Mechanics*, 638:243–266.

Contact: pratikprashant.aghor@unh.edu/ pratik.aghor54@gmail.com